VELOCITY PROFILE IN A TURBULENT BOUNDARY LAYER OF A SUPERSONIC GAS STREAM

A. D. Rekin

UDC 532.527.4:532.133

A semiempirical equation is proposed for relating the coefficients of molecular and turbulent viscosity in the laminar sublayer and in the transition region. Simple formulas are derived for determining the velocity throughout the boundary layer and the frictional stress at the surface of a plate at various values of the Mach number. The results of calculations are compared with experimental data.

Theoretical studies of the turbulent boundary layer which yield results applicable to engineering problems are, as a rule, based on the two-layer flow model. The Prandtl formula is used for frictional stress in the turbulent region and the Newton formula is applied to the predominantly viscous laminar sublayer. Such a two-layer model is, apparently, valid when the laminar sublayer is sufficiently thin. It has been shown in [1], on the basis of direct velocity measurements, that the thickness of the laminar sublayer increases greatly at a Mach number $Ma_\infty > 8$ and reaches 20-30% of the total boundary layer at $Ma_\infty = 9$-10. Consequently, the transitional layer between predominantly laminar flow and fully developed turbulent flow must also be appreciable. As the effect of molecular viscosity within the boundary layer extends over a wider region, the distribution of turbulent viscosity near the wall may strongly affect the profile of average velocities (and thus also of average temperature) across the boundary layer. Knowing the flow characteristics of the entire boundary layer is absolutely necessary for an analysis of the physicochemical processes in that layer.

We will first consider the boundary layer of an incompressible gas. The analysis will be based on the assumption that the tangential velocity of the gas is distributed as follows

$$\frac{u}{\sqrt{\tau_w/\rho}} = f(\eta), \quad \eta = \frac{\sqrt{\tau_w/\rho}\, y}{\nu} \tag{1}$$

over the entire boundary layer at a flat plate.

For small values of η (near the wall) function $f(\eta) = \eta$, for large values of η

$$f(\eta) = C\eta^{1/n}. \tag{2}$$

Coefficient C and the power exponent $1/n$ are weak functions of the Reynolds number [2]. Within the range $Re_x = 5 \cdot 10^5$-10^7, for example, they may be taken equal to 8.57 and 1/7 respectively (C = 8.57 rather than 8.74 in [2] agrees better with test data on skin friction at a plate). We will seek function $f(\eta)$ in the form

$$f(\eta) = \frac{\eta}{\left[1 + (a\eta)^k\right]^{\frac{n-1}{nk}}} \tag{3}$$

over the entire range of η, satisfying both extremes of small and large values. Here $a = (1/C)^{n/(n-1)}$. In addition to the known constants C and n, this expression contains also an exponent k which depends on the turbulent viscosity in the boundary layer. Indeed, the tangential friction stress in a turbulent stream is expressed as

$$\tau = \mu(1 + \varepsilon/\nu)(du/dy). \tag{4}$$

Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 23, No. 1, pp. 112-120, July, 1972. Original article submitted July 29, 1971.

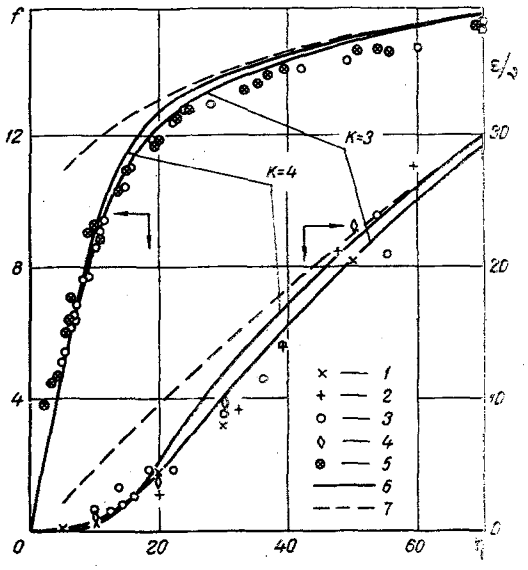

Fig. 1. Profile of average velocity and turbulent viscosity near a wall: 1) Schubauer data [4]; 2) Klebanov data [5]; 3) Simpson data [5]; 4) Laufer data [4]; 5) Conte-Bello data [6]; 6) relations (3) and (5); 7) relations (2) and (7).

Inserting (1) and (3) into (4), we obtain

$$1+\frac{\varepsilon}{\nu}=\frac{(\tau/\tau_w)}{df/d\eta}=\frac{(\tau/\tau_w)\left[1+(a\eta)^k\right]\left[1+(a\eta)^k\right]^{\frac{n-1}{nk}}}{1+(a\eta)^k/n}. \tag{5}$$

At small values of η ($\tau \approx \tau_w$), by expanding expression (5) into a series in terms of the small parameter ($a\eta$), one will find that

$$\varepsilon/\nu=(1-1/n)(1+1/k)(a\eta)^k. \tag{6}$$

At large values of η expression (5) also simplifies, to

$$\varepsilon/\nu=(\tau/\tau_w)(n/C)\,\eta^{1-1/n}. \tag{7}$$

The value for k will be selected on the basis of closest agreement between relations (3), (5), and test results. It follows from the flow equation and the continuity equation for a boundary layer [3] that $k \geq 3$ near the wall. In accordance with the various hypotheses, one usually lets k be equal to 3 or 4.

Results of calculations by formulas (3) and (5) have been plotted in Fig. 1 for $n = 7$, $C = 8.57$ ($a = 0.0814$), $k = 3$ and 4. On the same diagram is also shown the coefficient of turbulent viscosity based on test data by Schubauer, Klebanov, Simpson for a boundary layer and by Laufer for a channel. Data on the distribution of average velocity have been borrowed from recent studies concerning the boundary layer [5] and concerning the initial segment ($x/h < 40$) of a flat channel [6].

If the optimum value for k is to be selected on the basis of agreement with test results, then preference should be given to $k = 3$. This value has, therefore, been used for determining the function (3). Both f and ε/ν were also calculated with $k = 2$. The results here are found, however, to agree less closely with test results.

It is evident from Fig. 1 that the postulated relation (3) with $k = 3$ deviates from the linear relation $f = \eta$ when $\eta > 5$ and agrees with the power-law relation (2) when $\eta > 70$ (an evaluation of the power-law relation by comparison with test data at $\eta > 70$ can be found in any monograph on the turbulent boundary layer). We note that the dimensionless thickness of the transitional region between a viscous laminar flow and a fully developed turbulent flow ($\eta = 5$–70) is the same as suggested in [2].

An analysis of flow in the turbulent boundary layer of a compressible gas is based on an extension of relations (1) and (3), which have been verified experimentally for an incompressible gas. We assume that for a compressible gas the structure of relation (1) will not change when put in differential form

$$\frac{du}{\sqrt{\tau_w/\rho}}=df, \qquad d\eta=\frac{\sqrt{\tau_w/\rho}}{\nu}\,dy. \tag{8}$$

In integral form, relations (8) become

$$\frac{1}{\sqrt{\tau_w}}\int_0^u \sqrt{\rho}\,du = f(\eta),\ \eta = \sqrt{\tau_w}\int_0^y (\sqrt{\rho}/\mu)\,dy. \quad (9)$$

The values of the coefficients and the exponents in formula (3) for $f(\eta)$ will also be left unchanged. We note that, under these assumptions, expression (5) for the coefficient of turbulent viscosity will also remain unchanged.

The integral method, relations (9), of accounting for the variability of the physical properties of a medium was first used in the study [7] of heat transmission through a channel with water in the supercritical state, where its viscosity, density, and thermal conductivity decrease fast with rising temperature. The results of calculations agreed there closely with thermal flux measurements at the wall. In the case of a compressible gas, the viscosity and the density vary in the opposite sense as functions of the temperature. For this reason, the validity of postulating the relation (8) should be verified for various forms of the temperature-dependence of physical properties.

The general expression for frictional stress, according to the postulated relation (8) or (9) is

$$\tau_w = \frac{\left(\int_0^1 \sqrt{\rho/\rho_\infty}\,d\bar{u}\right)^2}{f^2(\eta_\delta)}\,\rho_\infty u_\infty^2. \quad (10)$$

We will consider the case of fully developed turbulence ($\eta_\delta > 70$), where the value of f can be determined from formula (2). Concerning the case where $\eta_\delta < 70$ and f must be determined from the general formula (3), we will comment appropriately as necessary.

Inserting the value $f = C\eta^{1/n}$ into (10) yields

$$\frac{C_f}{2} = \frac{\tau_w}{\rho_\infty u_\infty^2} = \left[\frac{n}{(n+1)(n+2)C^n}\right]^{\frac{2}{n+1}} \times \left(\frac{u_\infty \vartheta}{\nu_\infty}\right)^{-\frac{2}{n+1}} \left(\int_0^1 \sqrt{\rho/\rho_\infty}\,d\bar{u}\right)^{\frac{2n}{n+1}} \psi^{\frac{2}{n+1}}, \quad (11)$$

where

$$\vartheta = \int_0^\delta (\rho/\rho_\infty)\,\bar{u}\,(1-\bar{u})\,dy;$$

$$\psi = \frac{\int_0^1 (\rho/\rho_\infty)\,\bar{u}\,(1-\bar{u})\,d\bar{y}}{\frac{n}{(n+1)(n+2)}\int_0^1 \frac{\sqrt{\rho/\rho_\infty}}{\mu/\mu_\infty}\,d\bar{y}}. \quad (12)$$

For a boundary layer which is turbulent immediately at the front edge of the plate, Eq. (11) becomes

$$\frac{C_f}{2} = \left[\frac{n}{(n+2)(n+3)C^n}\right]^{\frac{2}{n+3}} \times \left(\frac{u_\infty x}{\nu_\infty}\right)^{-\frac{2}{n+3}} \left(\int_0^1 \sqrt{\rho/\rho_\infty}\,d\bar{u}\right)^{\frac{2n}{n+3}} \psi^{\frac{2}{n+3}}. \quad (13)$$

For an incompressible fluid with n = 7 (C = 8.57), (11) and (13) become the well-known formulas

$$\frac{C_{fi}}{2} = 0{,}013\left(\frac{u_\infty \vartheta}{\nu_\infty}\right)^{-0{,}25} = 0.0296\left(\frac{u_\infty x}{\nu_\infty}\right)^{-0{,}2}.$$

Calculations of function ψ have shown that its value in formulas (11) and (13) departs appreciably from unity. Thus, at a Mach number in the 0-15 range, at a Reynolds number in the $3\cdot10^6$-10^7 range, and at a temperature factor $t = T_W/T_r$ in the 0.1-1.0 range, for example, the value of ψ for air varies from 0.8 to 1.25. Therefore, in engineering calculations one may let ψ to the power $2/(n+1)$ be equal to 1.0 (or, for better precision, one may compute its value numerically on the basis of definition (12)). Thus, within the accuracy of the assumption that $\psi = 1$, the relative changes in the friction coefficient due to the compressibility of a gas flowing with Re_V = const and Re_x = const can be expressed simply and explicitly as

$$\left(\frac{C_f}{C_{fi}}\right)_\vartheta = \left(\int_0^1 \sqrt{\rho/\rho_\infty}\, d\bar{u}\right)^{\frac{2n}{n+1}}, \quad \left(\frac{C_f}{C_{fi}}\right)_x = \left(\int_0^1 \sqrt{\rho/\rho_\infty}\, d\bar{u}\right)^{\frac{2n}{n+3}}. \tag{14}$$

In order to calculate the friction coefficients by formula (14), one needs to have a relation between the gas density and the gas velocity in the boundary layer. The stipulation of constant pressure and specific heat across the boundary layer, in conjunction with the particular energy integral (the Crocco integral) corrected approximately for the restoration factor, yields the following well-known relation [8]:

$$\frac{\rho}{\rho_\infty} = \frac{T_\infty}{T} = \frac{1-\beta}{t + (1-t)\,\bar{u} - \beta \bar{u}^2}, \tag{15}$$

where $t = T_W/T_r$, $T_r = T_\infty(1+p)$ is the equilibrium wall temperature, $\beta = p/(1+p)$, $p = r(\varkappa - 1)Ma_\infty^2/2$, and r is the restoration factor usually assumed equal to 0.89 (relation (15) is exact when r = 1). When relation (15) is used, then the integrals (14) are taken in quadratures. As a result, we have

$$\left(\frac{C_f}{C_{fi}}\right)_\vartheta = \left[\sqrt{1/\beta - 1}\left(\arcsin\frac{1-t}{\sqrt{(1-t)^2 + 4t\beta}} + \arcsin\frac{2\beta - (1-t)}{\sqrt{(1-t)^2 + 4t\beta}}\right)\right]^{\frac{2n}{n+1}}, \tag{16}$$

$$\left(\frac{C_f}{C_{fi}}\right)_x = \left(\frac{C_f}{C_{fi}}\right)_\vartheta^{\frac{n+1}{n+3}}. \tag{17}$$

Formulas (16)-(17) simplify in special cases. When no heat transfer occurs (t = 1), for example,

$$(C_f/C_{fi})_\vartheta = \left[\sqrt{1/\beta - 1}\,\arcsin\sqrt{\beta}\right]^{\frac{2n}{n+1}},$$

or at a low gas velocity ($\beta \to 0$)

$$(C_f/C_{fi})_\vartheta = \left[2(1-\sqrt{t})/(1-t)\right]^{\frac{2n}{n+1}}.$$

According to these formulas, we find that the effect of the Reynolds number on the relative change in the friction coefficient is indirect (through the power exponent). The higher the Reynolds number is, the higher becomes the value of n and the stronger is the effect of gas compressibility. As $Re \to \infty$ ($n \to \infty$), the values of $\psi^{2/(n+1)}$ and $\psi^{2/(n+3)}$ in (11) and (13) become exactly equal to unity, and these formulas become the well-known relation at limiting conditions [8, 9]:

$$(C_f/C_{fi})_{\vartheta,x} = \left(\int_0^1 \sqrt{\rho/\rho_\infty}\, d\bar{u}\right)^2.$$

After the friction coefficient has been determined to the first approximation, one can find the velocity profile by numerically integrating Eq. (9). This equation can, for convenience, be rewritten as

$$\frac{1}{\sqrt{C_f/2}}\int_0^{\bar{u}} \sqrt{\rho/\rho_\infty}\, d\bar{u} = f\left(\sqrt{C_f/2}\,(u_\infty/\nu_\infty)\int_0^y \sqrt{\rho/\rho_\infty}/(\mu/\mu_\infty)\, dy\right).$$

Using relation (15), one first computes the left-hand side of Eq. (19) and then, with the aid of graphs representing relation (3) (Fig. 1), one finds successively η, y, η_δ, and δ. Equation (19) indicates that, unlike the frictional stress at wall, the velocity profile in the boundary layer is significantly affected by all parameters in (19).

Results of calculations according to formulas (16) and (17) with n = 7 are shown in Fig. 2 for the Mach number Ma ranging from 0 to 5 at Re_x = const and for the Mach number Ma ranging from 5 at Re = const. The dashed line corresponds to the limiting-condition formula (18) at t = 1. On the diagram are also shown the test results by Coles [11], Wilson [12], Matting et al. [13], Korkegi [14] without heat transfer at the wall

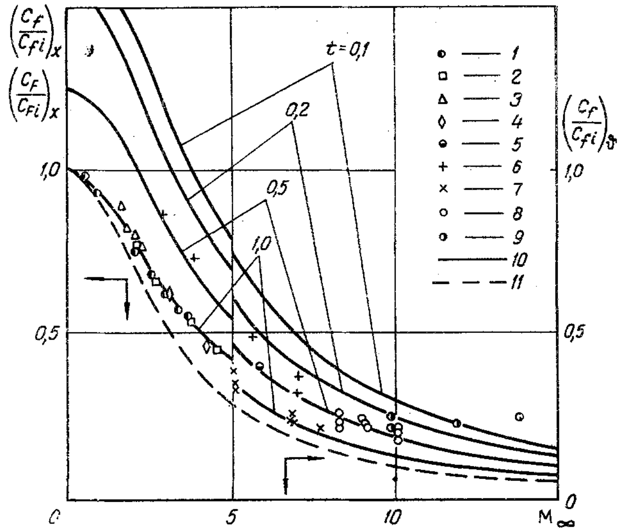

Fig. 2

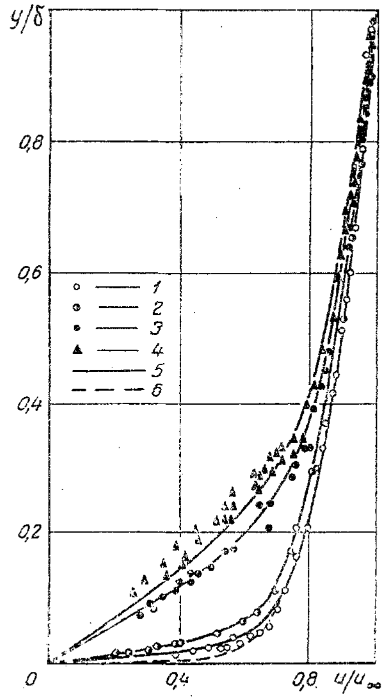

Fig. 3

Fig. 2. Relative variation of the friction coefficient with a changing Mach number: 1) data in [10]; 2) data in [11]; 3) data in [12]; 4) data in [13]; 5) data in [14]; 6) data in [15]; 7) data in [16]; 8) data in [1]; 9) data in [17]; 10) relations (16) and (17); 11) relation (18) with t = 1.

Fig. 3. Relative profile of average velocities in a turbulent boundary layer at various values of the Mach number: 1) data in [13] ($Ma_\infty = 4.2$, $Re_x = 6.2 \cdot 10^6$, t = 1); 2) data in [16] ($Ma_\infty = 6.83$, $Re_\vartheta = 8550$, t = 0.67); 3) data in [1] ($Ma_\infty = 9.07$, $Re_\vartheta = 2276$, t = 0.55); 4) data in [1] ($Ma_\infty = 10.04$, $Re_\vartheta = 1450$, t = 0.51); 5) calculations by formulas (8) and (3) for test conditions (1-4); 6) calculations by formulas (1) and (3) ($Ma_\infty \approx 0$, $Re_x = 10^7$, t = 1).

(t = 1), and Sommer and Short [15] (t = 0.18-0.43), Lobb et al. [16] (t = 0.5-0.67), Hill [1] (t = 0.47-0.53), and Nagamatsu et al. [17] (t = 0.214). According to Fig. 2, the calculations and the tests agree fairly up to $Ma_\infty = 10$. We note, furthermore that these calculations for Ma_∞ = 0-10 and t = 0.1-1.0 agree also within 12% with those in [8] and those based on the empirical formula in [18]:

$$(C_f/C_{fi})_x = t^{-0.27}\left[1 + r(\varkappa - 1)\,M_\infty^2/2\right]^{-0.55},$$

representing an approximation of test data. For Ma_∞ = 12 and 14, however, the test values from [17] shown in Fig. 2 are higher than calculated ones. One cause of this discrepancy is the incorrectness of calculations. Namely, the dimensionless thickness of the boundary layer η_δ calculated according to relation (19) for the given test conditions ($Re_\vartheta \approx 1000$, t = 0.214) is equal to 65 when $Ma_\infty = 12$ and equal to 32 when $Ma_\infty = 14$. This means, according to Fig. 1, that turbulence is not fully developed in the boundary layer. In this case, for a more accurate calculation of C_f/C_{fi}, one must insert into the denominator of (10) the value of $f(\eta_\delta)$ calculated according to (3), which at $\eta_\delta < 70$ is smaller than $C\eta_\delta^{1/n}$ (dashed lines in Fig. 1). Consequently, the calculated value of C_f/C_{fi} will be higher than shown in Fig. 2. As the Reynolds number Re_ϑ decreases at constant values of t and the Mach number, or as the Mach number Ma_∞ increases at constant values of t and the Reynolds number (i.e., as η_δ decreases), the effect of C_f/C_{fi} becomes increasingly stronger.

Some results of rather reliable velocity measurements in a turbulent boundary layer of air, with a laminar sublayer included in the test zone, are shown in Fig. 3. The test data here are by Matting et al. [13], Lobb et al. [16], and Hill et al. [1]. The solid lines represent calculation formulas (3) and (8) with n = 7 corresponding to the given test conditions. The dashed line represents calculated values for an in compressible fluid with $Re_x = 10^7$ (η_δ = 5000). The temperature-dependence of the air viscosity is based on the Sutherland formula. The calculated values of η_δ for tests 1-4 are respectively equal to 1460, 1580, 310, and 250, indicating a fully developed turbulence in the tested boundary layer.

A comparison between theoretical and experimental values in Fig. 3 will indicate a satisfactory agreement between them (the slight discrepancy between calculations and test values in the Hill experiment at $Ma_\infty = 10$ can be partly explained by an inefficient profile of the supersonic nozzle when used in these tests). On the basis of the values of the relative velocity gradients in the boundary layer, one can say that at higher values of the Mach number the laminar sublayer becomes a larger fraction of the turbulent boundary layer.

Calculations have shown that the relative thickness of the turbulent region of a boundary layer varies appreciably when the Reynolds number Re, the Mach number Ma_∞, and the temperature factor t vary, while the dimensionless thickness of the boundary layer η_δ varies within the 100-1000 range. If all test data in Fig. 3 as well as the data in [16] are plotted in coordinates $(\eta, \int_0^{\bar{u}} \sqrt{\rho/\rho_\infty}\, d\bar{u}/\sqrt{C_f/2})$, they will closely enough fit on a single curve (3).

NOTATION

x, y	are the coordinates along and across a plate;
u	is the average velocity not mean;
$\bar{u} = u/u_\infty$;	
$\bar{y} = y/\delta$;	
T	is the temperature;
ρ	is the density;
μ	is the dynamic viscosity;
ν, ε	are the coefficients of molecular and turbulent kinetic viscosity;
δ	is the boundary-layer thickness;
ϑ	is the momentum thickness;
Ma	is the Mach number;
Re	is the Reynolds number;
$t = T_W/T_r$	is the temperature factor;
τ	is the frictional stress;
C_f	is the friction coefficient;
η	is the dimensionless coordinate defined by formula (1) or (9);
f	is the function (3);
ψ	is the function (12);
n, C, a, k	are the coefficients in relations (2)-(7).

Subscripts and Superscripts

∞	denotes outside the boundary layer;
δ	denotes the edge of the boundary layer;
w	denotes wall;
i	denotes incompressible gas;
x, ϑ	denotes constant values of x and ϑ respectively.

LITERATURE CITED

1. F. Hill, Phys. Fluids, 2, No. 6 (1959).
2. G. Schlichting, Boundary-Layer Theory [Russian translation], Izd. IL (1956).
3. A. S. Monin and A. M. Yaglom, Statistical Hydromechanics [in Russian], Fizmatgiz, Moscow (1965), Pt. 1.
4. I. O. Khintse, Turbulence [in Russian], Fizmatgiz, Moscow (1963).
5. R. L. Simpson, J. Fluid Mech., 42, 4 (1970).
6. J. Conte-Bello, Turbulent Flow through a Channel with Parallel Walls [Russian translation], Izd. Mir, Moscow (1968).
7. K. Goldman, Nuclear Engineering, Part 1: Chemical Engineering Progress Symposium Series (1954).
8. Yu. V. Lapin, Turbulent Boundary Layer in Supersonic Gas Streams [in Russian], Izd. Nauka, Moscow (1954).

9. S. S. Kutateladze and A. I. Leont'ev, Turbulent Boundary Layer of a Compressible Gas [in Russian], Izd. Sibirsk. Otdel. Akad. Nauk SSSR, Novosibirsk (1962).
10. D. R. Chapman and R. M. Kester, J. Aeronaut. Sci., 20, No. 7 (1953).
11. D. Coles, J. Aeronaut. Sci., 21, No. 7 (1954).
12. R. E. Wilson, J. Aeronaut. Sci., 17, No. 9 (1950).
13. F. W. Matting, D. R. Chapman, J. R. Nyholm, and A. G. Thomas, NASA Rech. Report, 82 (1961).
14. R. M. Korkegi, J. Aeronaut. Sci., 23, No. 2 (1956).
15. S. C. Sommer and B. J. Short, J. Aeronaut. Sci., 23, No. 6 (1956).
16. R. K. Lobb, E. M. Winkler, and J. Persh, J. Aeronaut. Sci., 22, No. 1 (1955).
17. H. T. Nagamatsu, R. E. Sheer, and B. C. Graber, AIAA Paper No. 66-494 (1966).
18. V. S. Avduevskii et al., Principles of Heat Transfer in Aviation and Rocket Engineering [in Russian], Oborongiz (1960).